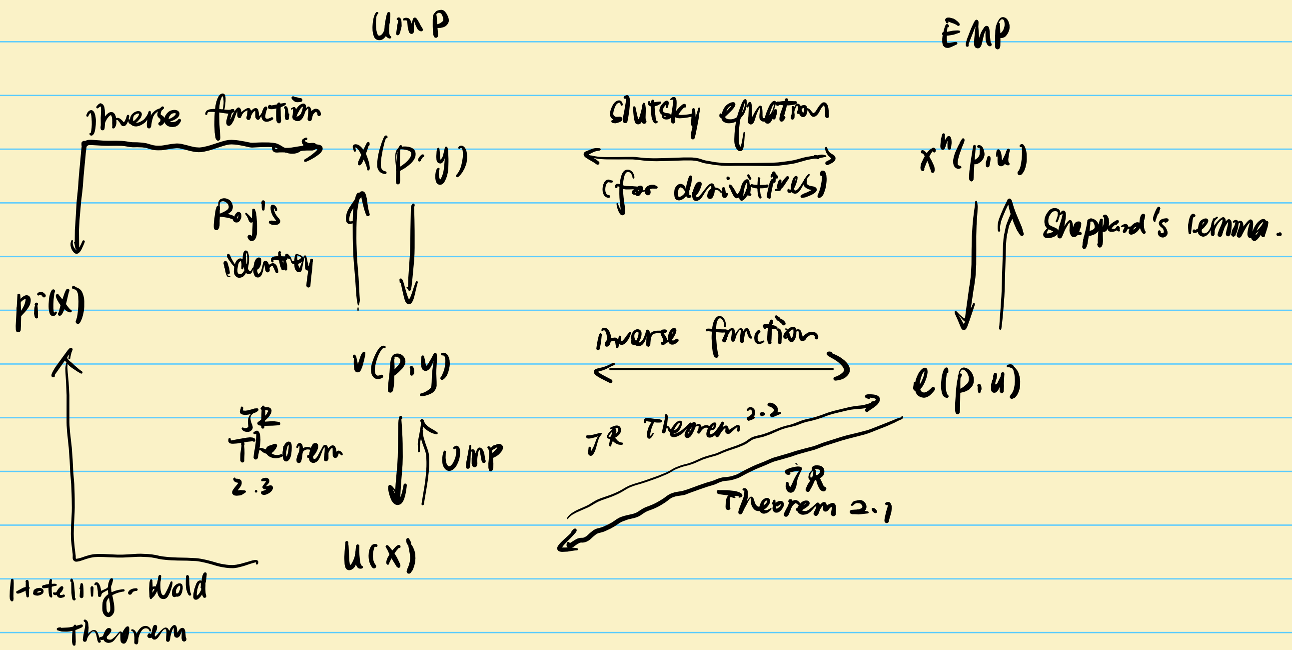
lab 3. Oct .4.

Duality problem



JR Theorem 2.1 Says: if $e(p,u)$ satisfies property 1-7.
(Then it is an expenditure function)
then we can construct a utility function $u(x)$
that is monotonic increasing and quasi-concave.

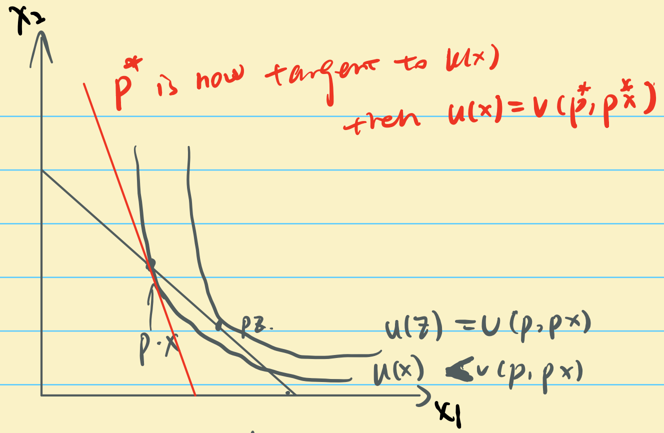
JR Theorem 2.2. The derived utility function $u(x)$ has the expenditure function $e(p,u)$.

JR Theorem 2.3.

$u(x) = \min_{p} v(p, px)$. How do we understand this?

for all the price level p. we always have $v(p.px) \geq u(x)$

This is because $v(p,px)$ answers a question about what is the maximized utility given a budget constraint px.



what makes $u(x) = v(p, px)$

we change the value of p

Hence $u(x) = \min_p v(p, px)$

And because of $v(p, y)$ being H.D. 0 over (p, y) we can normalize this minimization problem to

$$u(x) = \min_p v(p, 1) \quad \text{s.t.} \quad px = 1$$

Theorem 2.4 Hotelling-Wald Lemma.

The inverse demand function with income $y = 1$

$$p_i(x) = \frac{\frac{\partial u(x)}{\partial x_i}}{\sum_j x_j \frac{\partial u(x)}{\partial x_j}}$$

proof: since $u(x) = \min_p v(p, 1)$ s.t. $px = 1$

$$\mathcal{L} = v(p, 1) - \lambda(px - 1)$$

According to the envelope theorem.

$$\frac{\partial u}{\partial x_i} = \frac{\partial \mathcal{L}}{\partial x_i} = -\lambda p_i$$

Sum over i $\Rightarrow$ $\sum_j x_j \frac{\partial u}{\partial x_j} = -\lambda \underbrace{\sum p_j x_j}_{1} = -\lambda$

Hence $p_i(x) = -\frac{1}{\lambda} \cdot \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x_i} \Big/ \left(\sum_j x_j \frac{\partial u}{\partial x_j}\right)$

Integrability: if a marshallian demand $x(p,y)$ satisfies

① Walras' law (budget balance)

② Slutsky matrix symmetric

③ Slutsky matrix negative semi-definite

then $x(p,y)$ is utility-generated.

Questions

JR 2.6.

Question JR 2.6

A consumer has expenditure function $e(p_1, p_2, u) = up_1p_2/(p_1+p_2)$. Find a direct utility function, $u(x_1, x_2)$, that rationalizes this person's demand behavior.

We know $e(p_1, p_2, u)$ is an expenditure function
we are able to construct a $u(x)$

How? Inverse $\rightarrow$ $v(p, y) = \frac{p_1+p_2}{p_1p_2}$.

then $u(x) = \min_p v(p, 1)$ s.t. $p_1x_1+p_2x_2=1$.

JR.2.7. The use of Hotelling-wold lemma.

Question JR 2.7

Derive the consumer's inverse demand functions, $p_1(x_1, x_2)$ and $p_2(x_1, x_2)$, when the utility function is of the Cobb-Douglas form, $u(x_1, x_2) = Ax_1^{\alpha}x_2^{1-\alpha}$ for $0 < \alpha < 1$.

$$\frac{\partial u}{\partial x_1} = \alpha A x_1^{\alpha-1} x_2^{1-\alpha}$$

$$\frac{\partial u}{\partial x_2} = (1-\alpha) A x_1^{\alpha} x_2^{-\alpha}$$

$$x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} = Ax_1^{\alpha}x_2^{1-\alpha}$$

Hence. $p_1(x) = \frac{\alpha A x_1^{\alpha-1} x_2^{1-\alpha}}{A x_1^{\alpha} x_2^{1-\alpha}} = \frac{\alpha}{x_1}$

$p_2(x) = \frac{(1-\alpha)}{x_2}$

Example 2.3. Recover expenditure from consumers' demand

Suppose we have three goods.

$$x_i(p, y) = \frac{\alpha_i \cdot y}{p_i} \quad \text{where } i = 1, 2, 3. \ \sum \alpha_i = 1$$

It is easy to check this x. satisfies budget balance. symmetric and negative semi-definite slutsky matrix

Then this $x_i(p, y)$ is utility-generated.

The task is to find the solution $e(p, u)$ to

$$\frac{\partial e(p,u)}{\partial p_i} = \frac{\alpha_i e(p,u)}{p_i} \quad \text{(hicksian demand)}$$

$$\Rightarrow \frac{\partial \ln e(p,u)}{\partial p_i} = \frac{1}{e(p,u)} \frac{\partial e}{\partial p_i} = \frac{\alpha_i}{p_i}$$

$$\Rightarrow \ln e(p,u) = \alpha_i \ln p_i + c_1(p_{-i}, u)$$

This means that

$$\ln e(p,u) = \alpha_1 \ln p_1 + c_1 = \alpha_2 \ln p_2 + c_2 + \alpha_3 \ln p_3 + c_3$$

This implies

$$\ln e(p,u) = \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \alpha_3 \ln p_3 + c(u)$$

$$\Rightarrow e(p, u) = e^{c(u)} \cdot p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$$

we can let $e^{c(u)} = u$. or any other functional form that makes $e(p,u)$ monotonic increasing in u.

JR 2.5

indirect utility is $v(p, y) = y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}$

Now, verify the Roy's identity: $x_i(p,y) = -\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial y}}$

for x_1: $\quad x_1(p,y) = \dfrac{-(-\alpha_1 p_1^{-\alpha_1-1} p_2^{-\alpha_2} p_3^{-\alpha_3} y)}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}}$

$$x_1(p,y) = \frac{\alpha_1 y}{p_1}$$

$u(x) = \min\limits_{p} v(p, px) \quad$ s.t. $px = 1$

$$= \min_{p} \; p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3} - \lambda(p_1x_1 + p_2x_2 - p_3x_3 - 1)$$

Solving this we have

$$p_1^* = \frac{\alpha_1}{x_1} \qquad p_2^* = \frac{\alpha_2}{x_2} \qquad p_3^* = \frac{\alpha_3}{x_3}$$

then $u(x) = \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3}$, which is a Cobb-Douglas.

Question

Show that the following functions are homothetic.

(a) $y = \log x_1 + \log x_2$

(b) $y = e^{x_1 x_2}$

(c) $y = (x_1 x_2)^2 - x_1 x_2$

(d) $y = \log(x_1 x_2) + e^{x_1 x_2}$

(e) $y = \log\left(x_1^2 + x_1 x_2\right)^2$

whenever $f(\cdot)$ is first-order differentiable

$f(\cdot)$ is homothetic iff MRS is H.D.0

lets work on (c).

$$\frac{\partial y}{\partial x_1} = 2x_1 x_2^2 - x_2$$

$$\frac{\partial y}{\partial x_2} = 2x_1^2 x_2 - x_1$$

$\Rightarrow \text{MRS} = -\dfrac{2x_2(x_1x_2 - 1)}{2x_1(x_1x_2 - 1)} = \dfrac{x_2}{x_1}$. this is h.D.0 since $\dfrac{tx_2}{tx_1} = \dfrac{x_2}{x_1}$

So, $y = (x_1x_2)^2 - x_1x_2$ is a homothetic function.